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Convexity Preserving and Predicting by Bernstein Polynomials

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It is known that the Bernstein polynomials of a function f defined on $[0, 1]$ preserve its convexity properties, i.e., if $f^{(n)} \geqslant 0$ then for $m \geqslant n$, $(B_m f)^{(n)} \geqslant 0$. Moreover, if f is n-convex then $(B_m f)^{(n)} \geqslant 0$. While the converse is not true, we show that if f is bounded on (a, b) and if for every subinterval $[\alpha, \beta] \subset (a, b)$ the nth derivative of the mth Bernstein polynomial of f on $[\alpha, \beta]$ is nonnegative then f is n-convex.

It is known that the mth Bernstein polynomials preserve the n-convexity of f $(n \leqslant m)$. In this article we prove a weak converse theorem. We first recall the definition of n-convexity.

DEFINITION. A function f, defined on an interval I is said to be n-convex (on I) if the determinants

$$U(f; t_0, t_1, ..., t_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_n \\ t_0^2 & t_1^2 & \cdots & t_n^2 \\ \vdots & \vdots & & \vdots \\ t_0^{n-1} & t_1^{n-1} & \cdots & t_n^{n-1} \\ f(t_0) & f(t_1) & \cdots & f(t_n) \end{vmatrix} \geqslant 0, \tag{1}$$

whenever

$$t_0 < t_1 < \cdots < t_n \tag{2}$$

are $n + 1$ points of I.

If the points in (2) are equally spaced then the determinants (1) are denoted by

$$U_n(t_0, t_n; f). \tag{1'}$$

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Let f be defined on (a, b) and let $[\alpha, \beta] \subset (a, b)$. The mth Bernstein polynomial of f on $[\alpha, \beta]$ is defined by

$$B_m(f; [\alpha, \beta])(t) = 1/(\beta - \alpha)^m \sum_{k=0}^{m} \binom{m}{k} f(\alpha + kh)(t - \alpha)^k (\beta - t)^{m-k}, \quad (3)$$

where $h = (\beta - \alpha)/m$.

Theorem 1 shows that the mth Bernstein polynomials of f preserve its n-convexity (for $n \leqslant m$).

THEOREM 1 ([1, Theorem 6.3.3.]). *Let $f \in C([a, b])$. If f is n-convex then $B_m(f; [a, b])^{(n)} \geqslant 0$. In particular, if $f^{(n)} \geqslant 0$ then $B_m(f; [a, b])^{(n)} \geqslant 0$.*

The converse of this theorem is not true since $B_m(f; [a, b])$ is determined by the values f takes at $m + 1$ points. We shall, however, prove a weaker converse theorem which involves the mth Bernstein polynomials on all subintervals of (a, b).

LEMMA 1. *Let f be defined and bounded on (a, b) and let $\varepsilon > 0$. If the determinants $U_n(\alpha, \beta; f)$ are nonnegative for every interval $[\alpha, \beta] \subset (a, b)$ with $\beta - \alpha < \varepsilon$, then f is n-convex.*

The proof follows similar lines to those of Theorem 1 in [3].

THEOREM 2. *Let f be defined and bounded on (a, b) and let m and n be integers with $m \geqslant n$. Then f is n-convex iff $B_m(f; [\alpha, \beta])^{(n)} \geqslant 0$ on $[\alpha, \beta]$ (for all $[\alpha, \beta] \subset (a, b)$).*

Proof. The *only if* part is Theorem 1. To prove the *if* part, notice that

$$B_m(f; [\alpha, \beta])(t) = \sum_{k=0}^{m} (\Delta^{(k)} f)(\alpha) \binom{m}{k} \left(\frac{t - \alpha}{\beta - \alpha}\right)^k. \quad (4)$$

See [1, p. 108], where

$$(\Delta^k f)(\alpha) = h^k k!\, U_k(\alpha, k(\beta - \alpha)/m; f)/U_k(\alpha, k(\beta - \alpha)/m; u_k), \quad (5)$$

and where $u_k(t) = t^k$.

Since $B_m(f; [\alpha, \beta])^{(n)}(\alpha) \geqslant 0$, it follows from (4) and (5) that $U_n(\alpha, n(\beta - \alpha)/m; f) \geqslant 0$. If we consider intervals $[\alpha, \beta]$ with $\beta - \alpha < \varepsilon$ for some $0 < \varepsilon < b - a$, then by Lemma 1, f is n-convex on $(a, b - \varepsilon)$, and since we can choose ε arbitrarily small, f is n-convex on (a, b).

Remark. If, in addition, f is defined on $[a, b]$ and if it is continuous at a and b then f is n-convex on $[a, b]$. (The proof is similar to that of [2, Lemma 2].)

COROLLARY. *If for every* $[\alpha, \beta] \subset (a, b)$, $B_m(f; [\alpha, \beta])^{(n)}(\alpha) \geqslant 0$ *then* f *is n-convex and hence* $B_m(f; [\alpha, \beta])^{(n)} \geqslant 0$ *on* $[\alpha, \beta]$.

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